2D Linear Elasticity

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In the two dimensional elasticity problem we need to find the displacement $u = u_x\hat{e}_x + u_y\hat{e}_y$ at a point $(x, y) \in \Omega$, the domain. Extrapolation to two dimensions of the simplistic one dimensional strain $= \Delta l/l$, leads to a 2nd order strain tensor.

$$
\begin{aligned}
\epsilon &= \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} \end{bmatrix} \\
\epsilon_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}\right) && i, j \text{ are placeholders for } x, y \\
\implies \epsilon_{xx} &= \frac{\partial u_x}{\partial x} \\
\epsilon_{xy} &= \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)/2 \\
\epsilon_{yx} &= \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)/2 \\
\epsilon_{yy} &= \frac{\partial u_y}{\partial y}
\end{aligned}
$$

The constitutive relation is linear and is defined be a 4th order tensor C. So, the 2nd order stress tensor is given by

$$\sigma_{ij} = \sum_{k=x,y}\sum_{l=x,y} C_{ijkl}\epsilon_{kl}$$

where again i and j are placeholders for x or y. Let $\hat{n}$ be the outward normal on a point on the boundary of the domain Ω. Now, the components of the traction vector is given by

$$t_i = \sum_{j=x,y} \sigma_{ij} n_j \qquad\qquad i = x, y$$

The essential boundary condition would be to specify u, wheras the natural boundary condition would be to specify the traction t.

To make our lives easier we consider the engineering strain $[\epsilon_{xx}\,\epsilon_{yy}\,\gamma_{xy}]^\mathsf{T}$ where $\gamma_{xy} = 2\epsilon_{xy}$ is engineering shear strain. Now, we can write the constitutive relation using a matrix C.

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = C\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \gamma_{xy}\end{bmatrix}$$

Two of the usual choices for C are

$$\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}, \qquad \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}$$

where E is the Youngs modulus and ν is the Poisson's ratio.

Now that we have clearly defined the stress and strain we can write the governing equation

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + b_x = 0$$
$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + b_y = 0$$

where $b_x\hat{e}_x + b_y\hat{e}_y$ is the given body force at the point (x, y).